Quasirigidity: Some uniqueness issues

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(Received 21 March 2006; published 11 December 2006)

Quasirigidity means that one builds a theory for assemblies of grains under a slowly changing external load by using the deformation of those grains as a small parameter. Is quasirigidity a complete theory for these granular assemblies? Does it provide unique predictions of the assembly's behavior, or must some other process be invoked to decide between several possibilities? We provide evidence that quasirigidity is a complete theory by showing that two possible sources of indeterminacy do not exist for the case of disk-shaped grains. One possible source of indeterminacy arises from zero-frequency modes present in the packing. This problem can be solved by considering the conditions required to obtain force equilibrium. A second possible source of indeterminacy is the necessity to choose the status (sliding or nonsliding) at each contact. We show that only one choice is permitted, if contacts slide only when required by Coulomb friction.

DOI: 10.1103/PhysRevE.74.061303 PACS number(s): 45.70.−n, 81.05.Rm, 83.80.Fg

I. INTRODUCTION

A. Historical overview

The foundation of many physical theories is the observation that a certain physical quantity is "small." In practice, this means that the ratio between two different quantities with the same units is much less than unity. Once a small quantity has been identified, there are two ways of proceeding. First of all, that quantity can be set to zero, if one wishes to emphasize other aspects of the system. This is what is done when presenting the harmonic oscillator to students for the first time: one usually sets the dissipation to zero, even though its effects are quite important. The second possibility is to use the small quantity to linearize the equations. This is what one does when one suspects that the quantity, though small, plays an important role. An example is linear stability analysis.

In the study of granular materials, an obvious choice for a small quantity is the distance particles must move in order to activate contact forces. This choice is motivated by the common observation that, in a pile of stones or marbles, the deformation of the particles due to the stresses put on them is not visible to the naked eye. Should these deformations be set to zero, or kept as a small parameter?

At the end of the last century, this question was quite controversial, as one can see from browsing through a conference proceedings from that time [1]. The issue at hand was the stress distribution at the bottom of a sandpile. Several authors [2–5] proposed theories where the grains were assumed to be perfectly rigid. In this way, they could circumvent the question of a stress-free reference state. However, these theories were criticized on many points [6]. For example, it was pointed out that continuum mechanics could also account for the observations [7,8]. Another objection was that they used arbitrary ways to resolve the problem of contact force indeterminacy. This problem arises because it is impossible to deduce the contact forces in a static granular packing from assuming force equilibrium. There is no unique solution; instead many solutions are possible [9,10,12]. This loss of uniqueness occurs because there are more unknowns (contact forces) than equations (vanishing force and torque on each particle).

But the root of all these objections was the realization that rigid particle theories must be radically different from continuum mechanics. In continuum mechanics, the counterpart to the particle deformations is the strain. Thus neglecting deformations corresponds to eliminating the strain. But strain is a fundamental quantity, and eliminating it destroys the entire structure of continuum mechanics. Continuum mechanics can describe the macroscopic behavior of granular materials, and it would be quite strange if the best microscopic or grain-level theory were incompatible with it. It was no coincidence that most opposition to rigid particle theories came from engineers, who are more familiar with continuum mechanics than most physicists.

The recently proposed force network ensemble [11] can be considered as a modern version of the rigid particle theories. Instead of adding assumptions to determine the forces, all possibilities are considered. The system is thus represented by a point in a high-dimensional space [10,12], much as in statistical mechanics. One should also mention the widely used numerical method of contact dynamics [13], which is also based on the assumption of rigid particles.

This paper is concerned with an alternative approach, where the particle deformations are a small parameter of central importance. We call this approach "quasirigidity." This choice is motivated both by the belief that such deformations determine the contact forces in physical systems, and by the desire to propose a theory compatible with continuum mechanics, in view of arranging a future unification of microscopic and macroscopic theories.

B. Quasirigidity

Quasirigidity was first proposed as the basis for a numerical method [14–18], but has recently been explored theoreti-

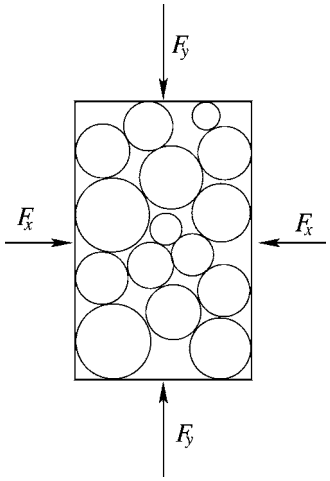

FIG. 1. Sketch of a biaxial test. An assembly of two-dimensional grains is confined by four walls. Forces F_x and F_y are applied to the vertical and horizontal walls.

cally by a number of authors [19–27]. Results of this approach include a deeper understanding of isostatic packings of frictionless particles [19,20,22], the microscopic origins of strain [19,21,22], the stability of packings [24,25], the relation between softening and sliding contacts [26], and jamming [27]. Work on a corresponding numerical method [21–23] has also continued.

In quasirigid theories, the state of the packing is given by the deformation of the particles at each contact. These deformations determine the contact forces, which in turn govern the motion of the particles. Finally, the particle motion gives the change of the deformations. When studying the response of a packing to an external load, these deformations must be given initial values, analogous to specifying a reference state in continuum mechanics.

Do these theories predict a unique evolution of the packing? Two possible sources of nonuniqueness have been pointed out [25]. First of all, indeterminacy can occur if there exists a possible motion that would not modify the contact forces. Such motions are called "floppy modes" or "mechanisms." We call this mechanism indeterminacy. Second, indeterminacy can arise due to the necessity of choosing the status (sliding or nonsliding) of each contact. This is contact status indeterminacy.

In this paper, we show that neither type of indeterminacy occurs. Mechanism instability appears to be a problem because force equilibrium is the usual starting point for quasirigid theories. But force equilibrium can be considered as a certain limit of Newton's second law. When this is done, one can assess the impact of any mechanisms that may be present. Contact status indeterminacy can be eliminated by requiring that the contacts obey Coulomb friction, and letting contacts slide only when necessary. This is what is commonly done in numerical simulations. A precise definition of "necessary" will be given later in this paper.

A third possible source of indeterminacy is opening or closing contacts. In this situation, one must consider transitions between four different statuses (open, nonsliding, and two different sliding directions). One would like to be sure that such transitions are always uniquely determined. Unfortunately, the methods developed in this paper are not sufficient to show this, so that this possible source of indeterminacy must be investigated in the future.

This paper is organized as follows. Section II presents an overview of the paper, detailing the questions posed in the Introduction, and sketching the results of the rest of the paper. Readers not wishing to savor the details may read this section, and then skip directly to Sec. V for a discussion of the results. The stiffness matrix for frictional disks is derived in Sec. III, including a discussion of mechanism indeterminacy in Sec. III E. Section IV deals with contact status indeterminacy by showing that there is a unique way to choose the contact status.

II. SYNOPSIS

A. The stiffness matrix

In this paper, we will deal with an assembly of disks, interacting via Coulomb friction and subjected to a slowly changing force. As a concrete example, consider a biaxial box, where a granular sample composed of disks is enclosed in a rectangular box of dimensions $L_x \times L_y$, with forces F_x and F_y exerted on the walls. These forces vary slowly with time, and one measures the resulting movement of the walls. A sketch of the biaxial box is shown in Fig. 1.

In Sec. III, we define the quasirigid limit precisely, and show that it leads to a piecewise linear behavior of the packing. Thus time can be divided into intervals $[t_i, t_{i+1}]$ during which the velocities of the particles are linearly related to the change in forces:

$$\frac{d\mathbf{f}_{\text{ext}}}{dt} = \mathbf{k}\mathbf{v}, \tag{1}$$

where $\mathbf{f}_{\text{ext}}$ represents the external forces (F_x and F_y for the biaxial box), $\mathbf{v}$ contains the velocities of the particles, and $\mathbf{k}$ is called the *stiffness matrix*.

The motion is only *piecewise* linear because the stiffness matrix $\mathbf{k}$ depends on the contact status. Whenever a contact status changes, therefore, $\mathbf{k}$ must be modified. Therefore, the times $\{t_i\}$ that define the intervals of linearity are the times when one or more contacts change status.

B. Indeterminacy of mechanism

What happens when the stiffness matrix has a zero eigenvalue? In that case, there exists $\mathbf{v}_* \neq \mathbf{0}$ such that $\mathbf{k}\mathbf{v}_* = \mathbf{0}$. Any multiple of $\mathbf{v}_*$ can be added to the solution of Eq. (1), and it would still be a solution. Thus it would seem that the theory is incapable of determining the amplitude of $\mathbf{v}_*$.

But it important to realize that one obtains Eq. (1) under the assumption that the external forces change on a time scale that is very long compared to the vibrations in the granular packing. The appearance of a zero eigenvalue corresponds to a diverging time scale of vibration, and thus the assumptions leading to Eq. (1) are not met. One must use instead Newton's second law, and in this case, the amplitude of $\mathbf{v}_*$ can be determined. If there is no interaction between the mechanism and the external force ($\mathbf{f}_{\text{ext}} \cdot \mathbf{v}_* = 0$), then the

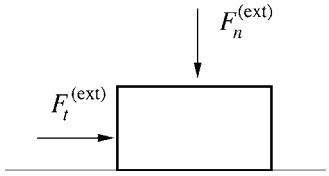

FIG. 2. A block pushed against a plane.

mechanism is decoupled from the other degrees of freedom, and Eq. (1) can still be used.

C. Contact status indeterminacy

Another source of indeterminacy may arise from the dependence of **k** on the contact status. Each contact in the packing may be sliding or nonsliding, and each choice leads to a different stiffness matrix. If there are M contacts, there are 2^M ways to assign contact status, and thus 2^M possible stiffness matrices, and 2^M different solutions to Eq. (1). Which one is correct? As stated in the Introduction, there is a unique solution if the contacts have Coulomb friction, and slide only when necessary. Coulomb friction means that the condition

$$\tilde{F} \equiv \mu F_n - |F_t| \geqslant 0 \tag{2}$$

must be obeyed at each contact. Here, F_n and F_t are the normal and tangential components of the contact force. The constant μ is the Coulomb friction coefficient.

When we say that contacts should slide only when necessary, we mean that they should slide only when they would violate Eq. (2) if they did not slide. Since this rule places an important role in this paper, we give it a name:

Principle of minimum sliding. A contact slides if, and only if, remaining nonsliding would violate Eq. (2).

To illustrate this principle, let us consider a block pushed against a plane with normal force $F_n^{(\mathrm{ext})}$, as shown in Fig. 2. A force tangential to the plane $F_t^{(\mathrm{ext})}$ is also applied. The contact between the block and the plane exerts a normal force $F_n^{(\mathrm{C})}$ and a tangential force $F_t^{(\mathrm{C})}$. Let these forces be directed opposite to the external ones, so $F_{n,t}^{(\mathrm{ext})} = F_{n,t}^{(\mathrm{C})}$ indicates force equilibrium. Let us suppose that $F_n^{(\mathrm{ext})}$ is fixed, while $F_t^{(\mathrm{ext})}$ is slowly increased from zero. As long as $F_t^{(\mathrm{ext})} \leqslant \mu F_n^{(\mathrm{ext})}$, we have $F_n^{(\mathrm{C})} = F_n^{(\mathrm{ext})}$ and $F_t^{(\mathrm{C})} = F_n^{(\mathrm{ext})}$, so the block remains in place, and the contact is nonsliding. When $F_t^{(\mathrm{ext})} > \mu F_n^{(\mathrm{ext})}$, the block must begin to slide, since the contact cannot cancel the imposed tangential force without violating Eq. (2). The contact is now sliding and $F_t^{(\mathrm{C})} = \mu F_n^{(ext)}$. Now suppose $F_t^{(\mathrm{ext})}$ is decreased again, so that $F_t^{(\mathrm{ext})} \leqslant \mu F_n^{(\mathrm{ext})}$. The block will deaccelerate, and finally stop. When the block stops, the principle of minimum sliding says that the contact must become nonsliding. Then the block will remain in place as long as $F_t^{(\mathrm{ext})} \leqslant \mu F_n^{(\mathrm{ext})}$. Note that if we did not apply the principle of minimum sliding, we would obtain a nonsensical result: the tangential contact force would remain constant at $\mu F_n^{(\mathrm{ext})}$ and start to accelerate the block. The frictional forces would thus be doing work on the block, which is a violation of the second law of thermodynamics. Thus, there is nothing artificial about the principle of minimum sliding. It is simply makes explicit what is needed to obtain sensible results. It also describes what is done in numerical simulations.

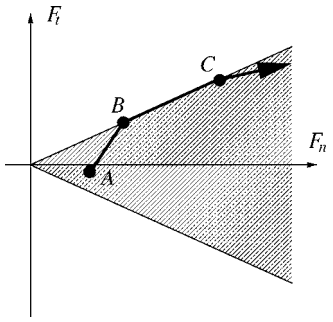

FIG. 3. The possible motion of a contact in the (F_n, F_t) plane. The Coulomb cone is shaded. The contact begins at point A within the cone. At B, it reaches the surface of the cone, becomes sliding, and moves along the cone boundary. At C it becomes nonsliding and moves into the cone interior.

Now let us return to the problem of choosing the contact statuses in a granular packing. It is helpful to consider the contact forces as points in the (F_n, F_t) plane. The set of forces (F_n, F_t) that obey Eq. (2) is shown in Fig. 3 as a shaded region. This set is called the *Coulomb cone* because it forms a cone when plotted this way.

As the particles move, the contact forces change, and thus trace out continuous trajectories in the (F_n, F_t) plane. These trajectories must of course remain within the Coulomb cone. For contacts in the interior of the Coulomb cone, any motion is allowed (for short enough times), since there is no danger they will leave the cone. Therefore, all such contacts will be nonsliding by the principle of minimum sliding.

Contacts whose forces lie on the boundary of the Coulomb cone $|F_t| = \mu F_n$ are called *critical* contacts, and must be handled carefully, since they may leave the Coulomb cone. If they are sliding, they will stay on the boundary. But as we saw above, we must also allow them to leave this surface and enter back into the interior of the Coulomb cone. Therefore, each critical contact could be sliding or nonsliding.

It seems that one could determine the status of the critical contacts simply by inspecting the particle velocities **v**. These velocities determine the change in contact forces, and one can easily determine if these changes would cause a critical contact to return to the interior of the Coulomb cone or not. However, changing a contact status also changes the matrix **k** and thus through Eq. (1) the velocities **v**. These new velocities may require changing the status of other contacts, provoking another recalculation of **v**, etc. Thus it seems one must use an iterative procedure. One assigns the contact status in a certain way, uses Eq. (1) to calculate the particle velocities **v**, and then begins to check if these velocities are consistent with the chosen statuses. If an inconsistency is found, the status must be changed, and the procedure begins again. This procedure must be continued until a solution is found.

In this paper, we are not concerned with this algorithm, but rather about the uniqueness and existence of a solution. Is it always possible to find a solution? Or are there many

solutions? Note that it is difficult to investigate these questions numerically. Even though we must only deal with the critical contacts, we are often faced with situations where all possibilities cannot be investigated. For example, it is common to have hundreds of critical contacts in numerical simulations involving thousands of particles. This means that the possible ways to choose the status cannot even be numbered with 64-bit integers.

Either nonexistence or nonuniqueness would bring up hard questions about the quasirigid approach. If there were sometimes no solutions, the theory could not be applied to those situations. On the other hand, if the solution were not unique, the stiffness matrix, combined with the principle of minimum sliding, would not be a complete description of the system. Some physical process must decide between the different possibilities. This unknown process would have been left out of the model, leading to indeterminacy in the same way that neglecting particle deformations leads to force indeterminacy. We would then have to ask what that physical process could be. One possibility is sound waves. As we show below, the quasistatic assumption amounts to removing "fast" processes like sound waves. When a contact changes status, there is probably a "negotiation" between the critical contacts, mediated by sound waves, that establishes their status. In the quasistatic limit, this period of negotiation becomes a single point in time, and it is assumed that the principle of minimum sliding suffices to determine the new status. Nonuniqueness of the choice of contact status means that the details of this negotiation must be taken into account.

D. Uniqueness of the solution

We now sketch the proof that that there is always one, and only one, choice of contact status that satisfies the principle of minimum sliding everywhere in the packing. We begin by defining some terms. Let the *state* of a packing be a way of assigning the status to all the critical contacts. To each state belongs a corresponding set of velocities $\mathbf{v}$, which can be calculated from Eq. (1). A state is *locally consistent at contact* α if the principle of minimum sliding is obeyed at that contact, and *locally inconsistent* otherwise. We also refer to the *consistency* of a contact, which means whether or not the principle of minimum sliding is obeyed there or not. A state is *globally consistent* if it is locally consistent at all contacts. We are thus concerned with the existence and uniqueness of the globally consistent state.

The proof has two premises. First, we assume that all possible states lead to a stable packing. The packing is stable if

$$\mathbf{v}^T\mathbf{k}\mathbf{v} > 0 \tag{3}$$

for a certain (large) class of relevant vectors $\mathbf{v}$. The second premise is the observation that the left-hand side of Eq. (1) is not modified by the status of the contacts. Thus if we consider two different states X and Y for the global contact status, one has

$$\frac{d\mathbf{f}_{\text{ext}}}{dt} = \mathbf{k}^X\mathbf{v}^X = \mathbf{k}^Y\mathbf{v}^Y, \tag{4}$$

where $\mathbf{k}^X$ is the stiffness matrix obtained if one chooses X, and $\mathbf{k}^Y$ is obtained by choosing Y. The corresponding velocities are $\mathbf{v}^X$ and $\mathbf{v}^Y$.

From Eqs. (3) and (4) it is possible to derive a series of inequalities, from which one may deduce the following theorem.

Status change theorem. If the status of any set of critical contacts changes, the consistency of at least one of those contacts must also change.

This statement is sufficient to prove both existence and uniqueness.

To show this, let us cast the theorem into a different form. A particular state corresponds to an M_c-bit binary number, $S \in \{0,1\}^{M_c}$, with each bit corresponding to the status of a single critical contact, and M_c is the number of critical contacts. For concreteness, let us say $S_\alpha = 1$ if contact α is sliding, and $S_\alpha = 0$ if it is nonsliding. To check the consistency of a given state S of contact status, we would construct the corresponding stiffness matrix $\mathbf{k}^S$, solve Eq. (1) for $\mathbf{v}^S$, and then check for consistency at each critical contact. The result of this procedure can be represented by a second M_c-bit binary number $C = \mathcal{C}(S)$, where each bit gives the consistency of a critical contact, i.e., $C_\alpha = 1$ if contact α is consistent, and $C_\alpha = 0$ otherwise. Now, the above theorem can be stated as follows: Changing any number of bits of S causes at least one of the corresponding bits in C to change. This means that no two different values of S can lead to the same value of C. There are 2^{M_c} possible choices for S, and 2^{M_c} possible values for C, so each possible value of C must be associated with a unique value of S. This applies also to $C = 111\ldots1$, corresponding to global consistency.

This result can be elegantly stated using a more mathematical language. The process of determining the consistency of a state defines a mapping $\mathcal{C}$ of one M_c-bit binary number to another:

$$\mathcal{C} : \{0,1\}^{M_c} \to \{0,1\}^{M_c},$$

$$S \mapsto \mathcal{C}(S). \tag{5}$$

Now the status change theorem sates simply that the mapping $\mathcal{C}$ is bijective. This proves the existence and uniqueness of the consistent choice since $\mathcal{C}$ is a mapping from $\{0,1\}^{M_c}$ onto itself.

III. THE STIFFNESS MATRIX

In this section, we present a derivation of Eq. (1), where the motion of the particles is related to the change in applied force by the stiffness matrix. Several related derivations have already been published [19,23,25,27]. The formulation presented here is distinguished from these other works in several ways. First, it incorporates sliding contacts, which is necessary to consider the uniqueness of the globally consistent contact status. The second difference is that Eq. (1) is shown to be a certain limit of Newton's second law. This is

essential to resolving the question of mechanism indeterminacy. On the other hand, the simplest possible form of the grains—disks in two dimensions—is assumed.

We first describe how interactions between the grains are modeled. We then assemble the quantities introduced here into vectors and matrices that describe all particles in the assembly. We then insert these results into Newton's second law to obtain equations for the motion of the grains. Then a limit of these equations is taken, leading to Eq. (1). A discussion of mechanism indeterminacy completes this section.

A. Particle interaction model

We suppose that the grains interact through cohesionless repulsion and Coulomb friction. We adopt the convention that a positive normal contact force F_n corresponds to repulsion. Thus the absence of cohesion requires

$$F_n \geqslant 0. \tag{6}$$

Recall that Coulomb friction means that Eq. (2) is obeyed.

When two grains first touch, two springs are created, one in the tangential and the other in the normal direction. The springs obey Hooke's law so that the normal and tangential contact forces F_n, F_t are proportional to the spring elongations D_n, D_t. To this restoring force, we add a linear damping to model the dissipation of energy:

$$F_n = -K_n D_n - \Gamma_n V_n, \quad F_t = -K_t D_t - \Gamma_t V_t, \tag{7}$$

where K_n and K_t are the spring constants, Γ_n and Γ_t are viscous damping coefficients, and V_n and V_t are the normal and tangential relative velocities. Here, $D_n < 0$ is interpreted as an overlap.

The springs are stretched by the relative motion of the particles, as long as this does not violate Eq. (2) or Eq. (6). When the contact is in the interior of the Coulomb cone, any motion is possible, so one has

$$\frac{dD_n}{dt} = V_n, \quad \frac{dD_t}{dt} = V_t, \tag{8}$$

where V_n and V_t are just the relative velocities at the point of contact:

$$V_n = (\vec{v}_i - \vec{v}_j) \cdot \hat{n}\hat{n},$$

$$V_t = (\vec{v}_i - \vec{v}_j) \cdot \hat{t}\hat{t} + r_i \omega_i + r_j \omega_j, \tag{9}$$

where $\vec{v}_i$, ω_i, and r_i are the velocity, angular velocity, and radius of particle i, and i and j label the touching particles. The vectors $\hat{n}$ and $\hat{t}$ are unit vectors pointing in the normal and tangential directions, respectively. Throughout this paper, capital letters indicate quantities concerning contacts, and small letters quantities concerning particles.

Now let us consider how to handle sliding contacts. It is helpful to define

$$\tilde{V} = \mu \frac{K_n}{K_t} V_n + V_t \operatorname{sgn} D_t. \tag{10}$$

Note that if the contact is nonsliding,

$$\frac{d\tilde{F}}{dt} = -(K_t + \Gamma_t)\tilde{V}, \tag{11}$$

where $\tilde{F}$ is defined in Eq. (2). For contacts on the boundary of the Coulomb cone, we have $\tilde{F}=0$. The sign of $\tilde{V}$ determines whether such contacts leave or remain within the Coulomb cone when made nonsliding. If $\tilde{V}<0$, the contact will move into the interior of the Coulomb cone ($\tilde{F}>0$). If $\tilde{V}>0$, the point would leave the Coulomb cone. The principle of minimum sliding can thus be reformulated.

Principle of minimum sliding. The status sliding is consistent if, and only if, $\tilde{V}>0$, where $\tilde{V}$ is defined in Eq. (10).

When a contact slides, Eq. (7) is still valid, but we set $\Gamma_t=0$ and constrain the spring elongations to change so that $\tilde{F}=0$. This can be accomplished if we use the first equation in Eq. (8) but replace the second with

$$\frac{dD_t}{dt} = -\left(\mu \frac{K_n}{K_t} \operatorname{sgn} D_t\right) V_n. \tag{12}$$

Once the contact forces are known, the net force $\vec{f}$ and torque τ on each particle can be computed:

$$\vec{f} = \sum_\alpha F_{\alpha,n}\hat{n}_\alpha + F_{\alpha,t}\hat{t}_\alpha,$$

$$\tau = r \sum_\alpha F_{\alpha,t}, \tag{13}$$

where the sums are taken over all the contacts that the concerned particle makes with its neighbors, and r is its radius.

B. Matrix formulation

It is useful to consider the preceding equations in matrix form. To do so, we must gather the various quantities into vectors. To begin with, we can group the force and torque exerted on particle i into a vector $\underline{f}_i$, and the contact forces exerted by a contact α into a vector $\underline{F}_\alpha$:

$$\underline{f}_i = \begin{pmatrix} f_{i,x} \\ f_{i,y} \\ \tau_i / r_i \end{pmatrix}, \quad \underline{F}_\alpha = \begin{pmatrix} F_{\alpha,n} \\ F_{\alpha,t} \end{pmatrix}. \tag{14}$$

It is often convenient to group these vectors together into high-dimensional quantities concerning all the particles or contacts in the packing:

$$\mathbf{f} = \begin{pmatrix} \underline{f}_1 \\ \underline{f}_2 \\ \vdots \\ \underline{f}_N \end{pmatrix} = \begin{pmatrix} f_{1,x} \\ f_{1,y} \\ \tau_1 / r_1 \\ \vdots \\ f_{N,x} \\ f_{N,y} \\ \tau_N / r_N \end{pmatrix} \tag{15}$$

and

$$\mathbf{F} = \begin{pmatrix} \underline{F}_1 \\ \underline{F}_2 \\ \vdots \\ \underline{F}_N \end{pmatrix} = \begin{pmatrix} F_{1,n} \\ F_{1,t} \\ \vdots \\ F_{M,n} \\ F_{M,t} \end{pmatrix}. \tag{16}$$

Here N is the number of bodies whose motion must be considered, and M is the number of contacts between these bodies. In these equations, and throughout this paper, boldface vectors will denote quantities concerning all contacts or particles (i.e., vectors in contact or particle space), whereas underscores indicate quantities associated with a single particle or contact.

Equation (13) can now be written

$$\underline{f}_i = \sum_{\alpha=1}^{M} \underline{c}_{i\alpha} \underline{F}_\alpha, \tag{17}$$

where $\underline{c}_{i\alpha}$ is a 3×2 matrix

$$\underline{c}_{i\alpha} = \begin{pmatrix} \chi_{i\alpha}\hat{n}_{\alpha x} & \chi_{i\alpha}\hat{t}_{\alpha x} \\ \chi_{i\alpha}\hat{n}_{\alpha y} & \chi_{i\alpha}\hat{t}_{\alpha y} \\ 0 & |\chi_{i\alpha}| \end{pmatrix}. \tag{18}$$

This gives the contribution of contact α to the force exerted on particle i. The symbol $\chi_{i\alpha}$ is defined as

$$\chi_{i\alpha} = \begin{cases} 1 & \text{if particle } i \text{ is first in contact } \alpha, \\ -1 & \text{if particle } i \text{ is second in contact } \alpha, \\ 0 & \text{if particle } i \text{ does not participate in contact } \alpha. \end{cases} \tag{19}$$

If a particle is "first" in contact α, that means that the contact exerts a normal force $F_{n,\alpha}\hat{n}_\alpha$ on it. If it is "second" in contact α, a normal force $-F_{n,\alpha}\hat{n}_\alpha$ is exerted on it. For each contact between two grains, one element of χ is 1, and another is -1. χ is also called the incidence matrix.

Equation (17) holds for each particle ($i=1,\ldots,N$). All of these equations can be written compactly using the definitions in Eqs. (15) and (16):

$$\mathbf{f} = \mathbf{c}\mathbf{F}. \tag{20}$$

The $3N\times 2M$ matrix $\mathbf{c}$ can be constructed by assembling an $N\times M$ array of the $\underline{c}_{i\alpha}$.

One can consider Eq. (20) as an equation for the unknown contact forces $\mathbf{F}$. However, one almost always has $2M > 3N$, meaning that $\mathbf{c}$ has at least $2M-3N$ linearly independent null eigenvectors. Therefore, Eq. (20) does not have a unique solution. This is the force indeterminacy problem discussed elsewhere in the literature [9–12,24].

We now continue by gathering the other quantities introduced in Sec. III A into vectors. Equation (7) can be written

$$\underline{F}_\alpha = -\underline{K}_\alpha \underline{D}_\alpha - \underline{\Gamma}_\alpha \underline{V}_\alpha \quad \text{or} \quad \mathbf{F} = -\mathbf{K}\mathbf{D} - \mathbf{\Gamma}\mathbf{V}, \tag{21}$$

where

TABLE I. Summary of stiffness matrix derivation as a chain of linear relations. The symbol $\propto$ is used to mean "is linearly related to."

Linear relation	Equation
Force on particles $\propto$ contact forces	$\mathbf{f}=\mathbf{c}\mathbf{F}$ (20)
Contact forces $\propto$ spring lengths	$\mathbf{F}=-\mathbf{K}\mathbf{D}$(21)
Change in spring length $\propto$ relative motion	$\dot{\mathbf{D}}=\mathbf{S}\mathbf{V}$ (25)
Relative motion $\propto$ particle motion	$\mathbf{V}=\mathbf{c}^T\mathbf{v}$ (23)

$$\underline{K}_\alpha = \begin{pmatrix} K_n & 0 \\ 0 & K_t \end{pmatrix}, \quad \underline{\Gamma}_\alpha = \begin{pmatrix} \Gamma_n & 0 \\ 0 & \Gamma_t \end{pmatrix}, \tag{22}$$

and $\mathbf{K}$ and $\mathbf{\Gamma}$ are $2M\times 2M$ diagonal matrices containing the $\underline{K}_\alpha$ or the $\underline{\Gamma}_\alpha$ on the diagonal.

Equation (9) can be written as

$$\underline{V}_\alpha = \sum_{i=1}^{N} (\underline{c}_{i\alpha})^T \underline{v}_i \quad \text{or} \quad \mathbf{V} = \mathbf{c}^T\mathbf{v}, \tag{23}$$

where $\mathbf{c}^T$ is the transpose of $\mathbf{c}$ [19]. Since the dimension of $\mathbf{V}$ is larger than that of $\mathbf{v}$, Eq. (23) places restrictions on $\mathbf{V}$. Not every vector $\mathbf{V}\in\mathbb{R}^{2M}$ is allowed, but only those vectors in the range of $\mathbf{c}^T$. Physically, this means that not every relative motion is possible, but only those that can be generated by moving and rotating the particles. The dimension of the range of $\mathbf{c}^T$ is at most $3N$. There are thus $2M-3N$ dimensions in $\mathbb{R}^{2M}$ that are inaccessible. These $2M-3N$ dimensions are precisely the null space of $\mathbf{c}$ [19].

Finally, the relation between $\mathbf{D}$ and $\mathbf{V}$ in Eqs. (8) and (12) requires careful treatment, due to the different possible contact statuses. Let $\mathbb{S}$ be the set of sliding contacts. We define a 2×2 matrix $\underline{S}_\alpha$ that depends on the status of contact α. If $\alpha\notin\mathbb{S}$, $\underline{S}_\alpha=\underline{1}$, if $\alpha\in\mathbb{S}$:

$$\underline{S}_\alpha = \begin{pmatrix} 1 & 0 \\ -\mu\dfrac{K_n}{K_t}\operatorname{sgn} D_t & 0 \end{pmatrix}. \tag{24}$$

Now the relation between $\mathbf{v}$ and $\mathbf{D}$ can be written

$$\frac{d\underline{D}_\alpha}{dt} = \underline{S}_\alpha \underline{V}_\alpha \quad \text{or} \quad \frac{d\mathbf{D}}{dt} = \mathbf{S}(\mathbb{S})\mathbf{V}. \tag{25}$$

$\mathbf{S}(\mathbb{S})$ is a block diagonal matrix, with the $\underline{S}_\alpha$ on the diagonal. It is a function of $\mathbb{S}$, as indicated.

Note that $\operatorname{sgn} D_t$ in Eq. (24) is a constant. In order for $\operatorname{sgn} D_t$ to change, the contact must cross the F_n axis in Fig. 3. This can only happen if the contact passes through the interior of the Coulomb cone. In that case, the contact would be nonsliding, and Eq. (24) would not be applied. The exception to this occurs when a contact approaches the origin. This brings up the question of what happens when a contact opens or closes. We are not dealing with that problem in this paper.

Our derivation of the global stiffness matrix is summarized in Table I. It rests on a chain of linear relations that can be established (or assumed) between the various quantities.

C. Equations of motion

At this point, most derivations of the stiffness matrix proceed directly to force equilibrium, and assume that the net forces $\mathbf{f}$ exerted on each particle are balanced by some externally imposed load $\mathbf{f}_{\text{ext}}$. We take a longer route that gives more insight into the situations where force equilibrium does not hold. We begin with Newton's second law, which relates the accelerations of the particles to the forces exerted on them:

$$\mathbf{m}\frac{d\mathbf{v}}{dt} = \mathbf{f} + \mathbf{f}_{\text{ext}}. \tag{26}$$

Here, $\mathbf{m}$ is a diagonal matrix containing the masses and momenta of inertia of all the grains. We could also write

$$\underline{m}_i \frac{d\underline{v}_i}{dt} = \underline{f}_i + \underline{f}_{\text{ext},i} \tag{27}$$

with

$$\underline{m}_i = \begin{pmatrix} m_i & 0 & 0 \\ 0 & m_i & 0 \\ 0 & 0 & I_i/r_i^2 \end{pmatrix}, \tag{28}$$

where m_i is the mass of particle i and I_i is its moment of inertia.

Combining Eq. (20), (21), and (26) gives

$$\mathbf{m}\frac{d\mathbf{v}}{dt} = -\mathbf{cKD} + \mathbf{f}_{\text{ext}}. \tag{29}$$

This equation can be differentiated once with respect to time, and Eqs. (23) and (25) can be used to obtain

$$\mathbf{m}\frac{d^2\mathbf{v}}{dt^2} = -\mathbf{cKSc}^T\mathbf{v} - \frac{d\mathbf{c}}{dt}\mathbf{KD} - \mathbf{c}^T\mathbf{\Gamma c}\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{f}_{\text{ext}}}{dt}. \tag{30}$$

The combination $\mathbf{cKSc}^T$ appears often, so we define $\mathbf{k}=\mathbf{cKSc}^T$ and write

$$\mathbf{m}\frac{d^2\mathbf{v}}{dt^2} = -\mathbf{kv} - \frac{d\mathbf{c}}{dt}\mathbf{KD} - \mathbf{c}^T\mathbf{\Gamma c}\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{f}_{\text{ext}}}{dt}. \tag{31}$$

This equation gives the full motion, without approximation, of the disks. Such an equation is solved numerically in the molecular dynamics simulation method. On the left-and side is the mass times the acceleration (differentiated by time), and on the right-hand side are the forces exerted on the particles (also differentiated by time).

D. Quasistatic balance

If one makes the quasirigid and quasistatic assumptions, then two terms dominate in Eq. (31). The quasirigid assumption means that the hardness of the particles is assumed to be much greater than the confining pressure, and the quasistatic assumption is that the external forces $\mathbf{f}_{\text{ext}}$ change much more slowly than any time scale associated with the contact forces. We describe the consequences of each of these assumptions below.

Let us begin with the quasirigid assumption. We will compare the first two terms on the right-hand side of Eq. (31). From Eq. (18), we see that $d\mathbf{c}/dt$ will be proportional to $d\hat{n}/dt$. Carrying out this differentiation,

$$\frac{d\hat{n}}{dt} = \frac{d}{dt}\left(\frac{\vec{x}_i - \vec{x}_j}{|\vec{x}_i - \vec{x}_j|}\right) = \frac{\vec{v}_i - \vec{v}_j}{|\vec{x}_i - \vec{x}_j|}\cdot\hat{t}\hat{t} \sim O\left(\frac{V}{R}\right). \tag{32}$$

Here, R is a typical particle radius, and V a typical relative velocity. Now we can estimate the sizes of the first two terms on the right-hand side of Eq. (31):

$$\mathbf{kv} = \mathbf{cKSV} \sim O(KV), \quad \frac{d\mathbf{c}}{dt}\mathbf{KD} \sim O\left(KV\frac{D}{R}\right), \tag{33}$$

where we have used $\mathbf{c}\sim\mathbf{S}\sim O(1)$. The quasirigid assumption implies that the deformations D are much smaller than a typical particle radius R, or $D/R \ll 1$. Thus the second term on the right-hand side is much smaller than the first, and Eq. (31) becomes

$$\mathbf{m}\frac{d^2\mathbf{v}}{dt^2} = -\mathbf{kv} - \mathbf{c}^T\mathbf{\Gamma c}\frac{d\mathbf{v}}{dt} + \frac{d\mathbf{f}_{\text{ext}}}{dt}. \tag{34}$$

The term that we have just neglected is called the "geometric stiffness" [25]. It can again become important in some situations. One example will be given in Sec. III E.

Now let us proceed to the quasistatic assumption. As stated above, this means that the external force $\mathbf{f}_{\text{ext}}$ is assumed to change much more slowly than any time scale associated with the vibrations in the granular assembly. To express the separation of time scales, we write

$$\frac{d}{dt} = \frac{d}{dt_0} + \epsilon\frac{d}{dt_1}, \tag{35}$$

where the variable t_0 measures a fast time scale, and t_1 a long time scale. In the problem at hand, the fast time scale describes vibrations of the packing, and the slow time scale is given by the change in the external load $\mathbf{f}_{\text{ext}}$. The presence of $\epsilon \ll 1$ before the derivative with respect to t_1 shows that these derivatives are small, as t_1 measures slow changes.

The assumption of quasistaticity can be expressed by saying that the particle positions $\mathbf{x}$ depend on both t_0 and t_1, but the external force $\mathbf{f}_{\text{ext}}$ depends only on the slow time t_1:

$$\mathbf{x} = \mathbf{x}_0(t_0) + \mathbf{x}_1(t_1), \quad \mathbf{f}_{\text{ext}} = \mathbf{f}_{\text{ext}}(t_1). \tag{36}$$

After differentiation by time, we have

$$\mathbf{v} = \mathbf{v}_0 + \epsilon\mathbf{v}_1 \tag{37}$$

and the $O(1)$ terms of Eq. (31) are

$$\mathbf{m}\frac{d^2\mathbf{v}_0}{dt_0^2} = -\mathbf{kv}_0 - \mathbf{c\Gamma c}^T\frac{d\mathbf{v}_0}{dt_0}. \tag{38}$$

This equation resembles that of a damped, harmonic oscillator, with three differences. First, it is an equation for the velocity, not the position. Second, it is a vector equation, not a scalar one. Finally, $\mathbf{k}$ and $\mathbf{\Gamma}$ depend on contact status. Nevertheless, it has the same properties as a damped, harmonic oscillator. The matrix $\mathbf{m}$ is positive definite and $\mathbf{c}^T\mathbf{\Gamma c}$ is non-negative definite, as $\mathbf{m}$ and $\mathbf{\Gamma}$ are both diagonal matrices with positive or non-negative entries. The stiffness matrix $\mathbf{k}$, however is not necessarily positive definite. If there are vectors

$\mathbf{v} \neq \mathbf{0}$ such that $\mathbf{v}^T\mathbf{k}v < 0$, then $\mathbf{k}$ acts like a negative number in Eq. (38), and $\mathbf{v}_0$ grows exponentially on a short time scale. This is "motion through an instability" [24]. Physically, the external forces push the particles *away* from the positions they must occupy in order to be in equilibrium [27]. In this case, one does not obtain the quasistatic balance; rather the packing is unstable and is set in motion.

On the other hand, if $\mathbf{v}^T\mathbf{k}v > 0$ then $\mathbf{k}$ acts like a positive number in Eq. (38), and $\mathbf{v}$ undergoes damped oscillations. In this case $\mathbf{v}_0 \to \mathbf{0}$ as $t_0 \to \infty$. One can then examine the $O(\epsilon)$ equations assuming $\mathbf{v}_0 = \mathbf{0}$. (The situation when $\mathbf{v}^T\mathbf{k}v = 0$ is more complicated, and will be discussed in the next section.) Keeping the $O(\epsilon)$ terms of Eq. (26) gives

$$\frac{d\mathbf{f}_{\text{ext}}}{dt} = \mathbf{k}\mathbf{v}_1. \tag{39}$$

This is the quasistatic balance, and the same as Eq. (1).

E. Mechanism indeterminacy

Now let us consider mechanism indeterminacy. This occurs when $\mathbf{k}$ has null eigenvalues, i.e., when $\mathbf{k}\mathbf{v}_* = 0$ for $\mathbf{v}_* \neq \mathbf{0}$. The amplitude of these motions cannot be determined from Eq. (39); thus they appear to be a source of indeterminacy, just as null eigenvalues of $\mathbf{c}$ [see Eq. (20)] indicate force indeterminacy in rigid particle theories. But there is one crucial difference between Eq. (20) and Eq. (39): the matrix $\mathbf{k}$ in Eq. (39) is a square matrix, so if it has null eigenvalue, its range has a lower dimension than the left-hand side of the equation. Thus there are external loads for which Eq. (39) has no solution. On the other hand, the range of $\mathbf{c}$ can have the same dimension as the right-hand side of Eq. (20), even when $\mathbf{c}$ has many null eigenvalues.

What happens when Eq. (39) has no solution? In that case, the quasistatic assumption used to derive Eq. (39) is invalid. This is because a null eigenvalue corresponds to a diverging period of vibration in the packing. Thus one cannot assume that the force is changing on a time scale much greater than the period of vibration. Therefore, one must consider Eq. (34) without assuming any separation of time scales. If one considers only the eigenvector $\mathbf{v}_*$, Eq. (34) can be integrated once to give

$$\mathbf{m}\frac{d\mathbf{v}_*}{dt} = -\mathbf{c}^T\boldsymbol{\Gamma}\mathbf{c}\mathbf{v}_* + \mathbf{f}_{\text{ext}}. \tag{40}$$

If the damping can be neglected, then $|\mathbf{v}_*| \propto t|\mathbf{f}_{\text{ext}}|/|\mathbf{m}|$. This is "motion through a mechanism" [24]. The growth of the velocity with time is much more gentle than for motion through an instability that was discussed in the previous section.

However, Eq. (39) can have a solution even when $\mathbf{k}$ has null eigenvalues. In this case, the imposed force does not excite the mechanisms. The mechanisms are irrelevant to the evolution of the system, and thus the derivation of Eq. (39) is again valid. Such irrelevant mechanisms are quite common. For example, if one constructs the stiffness matrix of the biaxial box discussed in Sec. II A, it is convenient to consider the walls as particles. Then, the stiffness matrix automatically has three null eigenvalues: two associated with the translation of the whole apparatus and one associated with its rotation. As long as the forces on opposite walls are equal, these modes will not be excited, and they are irrelevant. Another example is a "rattler"—a grain that has no contacts. Each one of its degrees of freedom yields a zero eigenvalue of $\mathbf{k}$, but it can be removed from the system without changing the quasistatic behavior.

Another thing that can happen when a mechanism is present is that motion can be stabilized or destabilized by the geometric stiffness $(d\mathbf{c}/dt)\mathbf{K}\mathbf{D}$ in Eq. (31), which was neglected in the quasirigid limit. When there is a mechanism, the particle displacements are no longer required to be small, so this term needs to be considered. An example is when two elliptical particles are pressed together [25]. If the particles are circular, there is a mechanism: the two particles can roll like bearings relative to one another. If the particles are made elliptical, the neglected term $(d\mathbf{c}/dt)\mathbf{K}\mathbf{D}$ must be considered to predict the behavior.

Thus, null eigenvalues of $\mathbf{k}$ can always be put into one of two classes. If Eq. (39) has no solution, the null eigenvalue signals the collapse of the packing, and the quasistatic assumption fails, and Eq. (34) must be used to predict the motion of the grains. On the other hand, if Eq. (39) still has a solution, even though the range of $\mathbf{k}$ has been reduced in dimension, the null eigenvalue corresponds to a degree of freedom that is irrelevant. Null eigenvalues appear to cause indeterminacy only because Eq. (39) is considered as the most fundamental equation. However, Eq. (39) is in fact an approximation to Eq. (34).

IV. CONTACT STATUS INDETERMINACY

In this section, we consider contact status indeterminacy. We will show that only one state does not lead to a violation of the principle of minimum sliding at one or more contacts. To do so, we must compare the stiffness matrices of the different states. We begin by presenting the two hypotheses needed for the proof: first, that all possible states are stable, and second that the applied load is independent of the state. Then we consider an example where only two contacts are critical, and show that the consistent state always exists and is unique. Then we consider the general case with an arbitrary number of critical contacts.

A. Conditions needed to show uniqueness

1. The stability condition

A packing is stable if the quadratic form $Q = \mathbf{v}^T\mathbf{k}\mathbf{v}$ is positive [24,25,27]:

$$Q(\mathbf{v}, \mathbb{S}) = \mathbf{v}^T\mathbf{k}(\mathbb{S})\mathbf{v} > 0, \tag{41}$$

where $\mathbb{S}$ is the set of contacts that are sliding. As we saw in Sec. III D, the motion cannot be assumed to be quasistatic when $Q \leqslant 0$.

The quadratic form plays an important role in this paper, so we will discuss how it can be calculated. If we recall the definition $\mathbf{k} = \mathbf{c}\mathbf{K}\mathbf{S}\mathbf{c}^T$, and group factors in a suggestive way, we have

$$Q(\mathbf{v},\mathbb{S}) = [\mathbf{v}^T\mathbf{c}]^T\mathbf{KS}[\mathbf{c}^T\mathbf{v}] = \mathbf{V}^T[\mathbf{KS}]\mathbf{V} \tag{42}$$

The matrix **KS** is block diagonal, with each block corresponding to a contact. Thus $Q(\mathbf{v},\mathbb{S})$ reveals itself to be simply a sum over contacts:

$$Q(\mathbf{v},\mathbb{S}) = \sum_{\alpha=1}^{M} \underline{V}_\alpha^T \underline{K}_\alpha \underline{S}_\alpha \underline{V}_\alpha,$$

$$= \sum_{\alpha \notin \mathbb{S}} Q_\alpha^{(\mathrm{NS})} + \sum_{\alpha \in \mathbb{S}} Q_\alpha^{(\mathrm{S})}, \tag{43}$$

where $Q_\alpha^{(\mathrm{NS})}$ is the contribution of contact α if it is nonsliding, and $Q_\alpha^{(\mathrm{S})}$ is its contribution if it is sliding. Using Eqs. (22) and (24), we have

$$Q_\alpha^{\mathrm{NS}}(\mathbf{v}) = K_n V_{n,\alpha}^2 + K_t V_{t,\alpha}^2, \tag{44}$$

$$Q_\alpha^{(\mathrm{S})}(\mathbf{v}) = K_n V_{n,\alpha}^2 - \mu K_n V_{t,\alpha} V_{n,\alpha} \operatorname{sgn} D_{t,\alpha}. \tag{45}$$

In the following, it is useful to use Eq. (10) and replace $V_{n,\alpha}$ with $\tilde{V}_\alpha$. Equation (45) becomes

$$Q_\alpha^{(\mathrm{S})}(\mathbf{v}) = Q_\alpha^{(\mathrm{NS})}(\mathbf{v}) - K_t V_{t,\alpha} \tilde{V}_\alpha \operatorname{sgn} D_{t,\alpha}. \tag{46}$$

Now let us define

$$\hat{F}_\alpha = K_t V_{t,\alpha} \operatorname{sgn} D_{t,\alpha}, \tag{47}$$

so that Eq. (46) becomes

$$Q_\alpha^{(\mathrm{S})}(\mathbf{v}) = Q_\alpha^{(\mathrm{NS})} - \hat{F}_\alpha \tilde{V}_\alpha. \tag{48}$$

Therefore, the stability condition Eq. (41) is

$$Q(\mathbf{v},\mathbb{S}) = Q(\mathbf{v},\varnothing) - \sum_{\alpha \in \mathbb{S}} \hat{F}_\alpha \tilde{V}_\alpha > 0. \tag{49}$$

Note that $Q(\mathbf{v},\varnothing) > 0$, because the contribution of each contact must be positive. This means the only way to obtain an unstable packing is for the sliding contacts to make large and negative contributions to Q.

In the following, it will be necessary to compare Q for different states. If the sliding contacts present in a given state are divided into two disjoint sets $\mathbb{S}_1$ and $\mathbb{S}_2$ ($\mathbb{S}_1 \cap \mathbb{S}_2 = \varnothing$), then

$$Q(\mathbf{v},\mathbb{S}_1 \cup \mathbb{S}_2) = Q(\mathbf{v},\mathbb{S}_1) - \sum_{\alpha \in \mathbb{S}_2} \hat{F}_\alpha \tilde{V}_\alpha. \tag{50}$$

2. The independent load condition

Let us consider two different states X and Y, each with a different set of sliding contacts. Let $\mathbb{S}$ be the set of contacts sliding in both states, $\mathbb{X}$ be set of sliding contacts unique to X and $\mathbb{Y}$ be those unique to Y (see Table II). Let $\mathbf{v}^X$ be the velocities in state X and $\mathbf{v}^Y$ be those in Y. Similarly, $\mathbf{k}^X = \mathbf{k}(\mathbb{S} \cup \mathbb{X})$ and $\mathbf{k}^Y = \mathbf{k}(\mathbb{S} \cup \mathbb{Y})$. If the externally applied force is independent of contact status,

$$\frac{d\mathbf{f}_{\mathrm{ext}}}{dt} = \mathbf{k}^X\mathbf{v}^X = \mathbf{k}^Y\mathbf{v}^Y, \tag{51}$$

or both $\mathbf{v}^X$ and $\mathbf{v}^Y$ are the velocities caused by the same external forces, but with different stiffness matrices.

TABLE II. The two states X and Y considered in the independent load condition. The status of the contacts in the sets $\mathbb{X}$, $\mathbb{Y}$, and $\mathbb{S}$ is given (S=sliding, NS=nonsliding). All contacts not in these sets are nonsliding in both states.

	State			
Status	$\mathbb{X}$	$\mathbb{Y}$	$\mathbb{S}$	All others
X	S	NS	S	NS
Y	NS	S	S	NS

Equation (51) can be rewritten

$$\mathbf{k}^X\mathbf{v}^X - \mathbf{k}^Y\mathbf{v}^Y = \mathbf{cK}[\mathbf{S}^X\mathbf{V}^X - \mathbf{S}^Y\mathbf{V}^Y] = 0. \tag{52}$$

Now let us multiply this equation from the left by $(\mathbf{v}^X - \mathbf{v}^Y)^T$:

$$[\mathbf{V}^X - \mathbf{V}^Y]^T\mathbf{K}[\mathbf{S}^X\mathbf{V}^X - \mathbf{S}^Y\mathbf{V}^Y] = 0. \tag{53}$$

This again is simply a sum over contacts:

$$\sum_{\alpha=1}^{M} [\underline{V}_\alpha^X - \underline{V}_\alpha^Y]^T \underline{K}_\alpha [\underline{S}_\alpha^X \underline{V}_\alpha^X - \underline{S}_\alpha^Y \underline{V}_\alpha^Y] = 0. \tag{54}$$

There will be four types of contributions, corresponding to the four columns in Table II: (1) Contacts that slide in X but not in Y (the set $\mathbb{X}$), (2) contacts that slide in Y but not in X (the set $\mathbb{Y}$), (3) contacts that are sliding in both X and Y (the set $\mathbb{S}$); and (4) contacts that are nonsliding in both X and Y.

For the last two classes of contacts, $\underline{S}^X = \underline{S}^Y$, so their contributions here will be the same as to the quadratic form. For contacts $\alpha \in \mathbb{X}$ the contribution is

$$K_n(V_{n,\alpha}^X - V_{n,\alpha}^y)^2 + (\mu K_n V_{n,\alpha}^X \operatorname{sgn} D_{t,\alpha} + K_t V_{t,\alpha}^Y)(V_{t,\alpha}^X - V_{t,\alpha}^y). \tag{55}$$

Defining $\hat{F}_\alpha^{XY} = \hat{F}_\alpha^X - \hat{F}_\alpha^Y$, this quantity can be rewritten as

$$Q_\alpha^{(\mathrm{NS})}(\mathbf{v}^X - \mathbf{v}^Y) - \hat{F}_\alpha^{XY} \tilde{V}_\alpha^X. \tag{56}$$

In the same way, the contribution of contacts $\alpha \in \mathbb{Y}$ is

$$Q_\alpha^{(\mathrm{NS})}(\mathbf{v}^X - \mathbf{v}^Y) + \hat{F}_\alpha^{XY} \tilde{V}_\alpha^Y. \tag{57}$$

Thus Eq. (53) becomes

$$Q(\mathbf{v}^X - \mathbf{v}^Y,\mathbb{S}) = \sum_{\alpha \in \mathbb{X}} \hat{F}_\alpha^{XY} \tilde{V}_\alpha^X - \sum_{\alpha \in \mathbb{Y}} \hat{F}_\alpha^{XY} \tilde{V}_\alpha^Y. \tag{58}$$

B. Small numbers of sliding contacts

In preparation for treating the general case, we will consider the problem of a packing that may slide at two different contacts β and γ. The four different possible states are shown in Table III and labeled A, B, C, and D. We will use superscripts to indicate quantities belonging to each state. For example $\mathbf{v}^A$ are the particle velocities in state A and $\mathbf{S}^C$ is the status matrix in state C.

The system starts in state A with no sliding contacts. Then contact β reaches the boundary of the Coulomb cone and

TABLE III. The states considered in Sec. IV B. Contacts β and γ are critical, so there are four possible states labeled by A, B, C, and D. The table gives the status of each critical contact in each state (S=sliding, NS=nonsliding). Also given are the signs of $\tilde{V}$. $\tilde{V}^A_\beta$ and $\tilde{V}^B_\gamma$ are known to be positive, because it is assumed that β becomes critical while the packing is in state A, and γ becomes critical while in state B. The other values are deduced from the one contact status change theorem given at the end of Sec. IV B 1. $\sigma_1, \sigma_2 = \pm 1$ are unknown, but used here to show relations between different states.

State	Contact status		sgn $\tilde{V}$		Consistency requirements
	β	γ	β	γ	
A	NS	NS	1	σ_2	β, γ not critical
B	S	NS	1	1	γ not critical
C	S	S	σ_1	1	$\sigma_1 = 1$
D	NS	S	σ_1	σ_2	$\sigma_1 = -1$, $\sigma_2 = 1$

becomes sliding, and the packing moves to state B. Then contact γ reaches the boundary, the system moves to either state C, where both β and γ slide, or to state D, where only γ slides.

We will consider the questions of existence and uniqueness. For example, when β becomes sliding, is it guaranteed that B is consistent? If it were inconsistent, then contact β should become nonsliding. But if it became nonsliding, the system would return to state A, and β would leave the Coulomb cone. A solution would not exist. The question of uniqueness arises when contact γ reaches the boundary. If both states C and D were consistent, the system could move to either C or D, and the solution would not be unique. We will show that the consistent state exists and is unique.

1. One sliding contact

We first consider the transition from A to B. This transition occurs when the contact β reaches the boundary of the Coulomb cone. It starts somewhere within the cone, that is with $\tilde{F}_\beta > 0$ [see Eq. (2)]. As the contact moves toward the boundary, $\tilde{F}_\beta$ decreases and then vanishes when β reaches the boundary. Therefore, Eq. (11) requires that

$$\tilde{V}^A_\beta > 0. \tag{59}$$

Thus a 1 is given in the top row of the third column of Table III.

One usually supposes without comment that the state B is consistent. But this is not obvious, because all particles change their velocities when the state changes. The state B will be consistent only if $\tilde{V}^B_\beta > 0$, and no one has shown that this must be so.

To show that B is indeed consistent, let us apply the independent load condition to the transition between A and B. Setting $X=A$, $Y=B$, $\mathbb{S}=\mathbb{X}=\emptyset$, and $\mathbb{Y}=\{\beta\}$, Eq. (58) becomes

$$Q(\mathbf{v}^A - \mathbf{v}^B, \emptyset) = -\hat{F}^{AB}_\beta \tilde{V}^B_\beta. \tag{60}$$

If state A is stable, the quadratic form must be positive, leading to

$$\hat{F}^{AB}_\beta \tilde{V}^B_\beta < 0, \tag{61}$$

which gives us some information about the sign of $\tilde{V}^B_\beta$, but also unfortunately involves the unknown quantity $\hat{F}^{AB}_\beta$. More information can be obtained by requiring state B to be stable:

$$Q(\mathbf{v}^A - \mathbf{v}^B, \{\beta\}) > 0, \tag{62}$$

and after using Eq. (50)

$$Q(\mathbf{v}^A - \mathbf{v}^B, \emptyset) - \hat{F}^{AB}_\beta (\tilde{V}^A_\beta - \tilde{V}^B_\beta) > 0, \tag{63}$$

and finally using Eq. (60):

$$\hat{F}^{AB}_\beta \tilde{V}^A_\beta < 0. \tag{64}$$

Together Eqs. (64) and (61) show that $\tilde{V}^A_\beta$ and $\tilde{V}^B_\beta$ have the same sign. We already showed that $\tilde{V}^A_\beta > 0$ in Eq. (59); thus $\tilde{V}^B_\beta > 0$ as well. Therefore state B is compatible, and the solution exists.

Before proceeding, let us pause to note that the reasoning we have just employed does not depend on state A being without sliding contacts. Define $\mathbb{S}$ to be the set of contacts sliding in state A. If we simply replace the empty set in Eqs. (60) and (63) with $\mathbb{S}$, and $\{\beta\}$ with $\{\beta\} \cup \mathbb{S}$ in Eq. (62), the reasoning remains unchanged. Thus we have a general statement: If two states A and B differ only in the status of a single contact β, then sgn $\tilde{V}^A_\beta$=sgn $\tilde{V}^B_\beta$. This state implies that the consistency at contact β must be different in states A and B. If sgn $\tilde{V}^A_\beta$=sgn $\tilde{V}^B_\beta$=-1 (or if β is not a critical contact) then A will be consistent at β and B will be inconsistent. If sgn $\tilde{V}^A_\beta$=sgn $\tilde{V}^B_\beta$=1 then A will be inconsistent and B consistent. Thus we have proved a special case of the status change theorem.

One-contact status change theorem. If two states A and B differ only in the status of a single contact β, then sgn $\tilde{V}^A_\beta$ =sgn $\tilde{V}^B_\beta$, meaning that they differ in the consistency at β.

Using this theorem, we can now fill in the third and fourth columns of Table III. States C and D differ only in the status of contact β, so sgn $\tilde{V}^C_\beta$=sgn $\tilde{V}^D_\beta = \sigma_1$. States A and D differ only in the status of contact γ, so sgn $\tilde{V}^A_\gamma$=sgn $\tilde{V}^D_\gamma = \sigma_2$. For the same reason, sgn $\tilde{V}^B_\gamma$=sgn $\tilde{V}^C_\gamma$, and in the next section, we show sgn $\tilde{V}^B_\gamma$=1.

2. Two sliding contacts

Suppose now that the system is in state B, when a second contact γ reaches the boundary of the Coulomb condition. Following the same reasoning as at the beginning of the preceding Sec. IV B 1, we see that

$$\tilde{V}^B_\gamma > 0. \tag{65}$$

Thus a 1 is given in the second row of the fourth column of Table III. The one-contact status change theorem tells us we

should immediately put a 1 in the third row of the same column.

Now let us consider the uniqueness of the globally consistent state. When γ becomes sliding, the system can now move to either state C where both β and γ slide, or to D, where only γ slides.

The one-contact status change theorem shows that the solution is unique. Only a single contact is different between states C and D, and the theorem states that only one of these states can be consistent at β. If $\operatorname{sgn} \tilde{V}_\beta^C = \operatorname{sgn} \tilde{V}_\beta^D = \sigma_1 = 1$, then contact β must slide, and C is consistent but not D. On the other hand, if $\sigma_1 = -1$, then contact β must be nonsliding, and D is consistent but not C.

Now let us check the existence of the solution. Suppose $\sigma_1 = 1$, meaning the system must move to state C. In order for this state to be compatible, we must also have $\tilde{V}_\gamma^C > 0$. As shown above, the one-contact status change theorem guarantees this.

What happens if $\sigma_1 = -1$? In this case, the system must move to state D, but is this state compatible? The first-order state theorem says $\operatorname{sgn} \tilde{V}_\gamma^D = \operatorname{sgn} \tilde{V}_\gamma^A = \sigma_2$, but this does not help prove compatibility, since σ_2 is unknown. However, we do know that $\tilde{V}_\gamma^B > 0$, so let us now search for a way to relate $\tilde{V}_\gamma^B$ to $\tilde{V}_\gamma^D$. This cannot be done by the one-contact status change theorem, because B and D differ at two contacts. Therefore, we must return to two hypotheses discussed in Sec. IV A, and deduce more information from them.

If we apply the independent load condition to states B and D, we obtain

$$Q(\mathbf{v}^B - \mathbf{v}^D, \varnothing) = \hat{F}_\beta^{BD} \tilde{V}_\beta^B - \hat{F}_\gamma^{BD} \tilde{V}_\gamma^D. \tag{66}$$

Requiring all four states A, B, C, and D to be stable leads to the inequalities

$$\hat{F}_\beta^{BD} \tilde{V}_\beta^B > \hat{F}_\gamma^{BD} \tilde{V}_\gamma^D, \tag{67}$$

$$\hat{F}_\beta^{BD} \tilde{V}_\beta^D > \hat{F}_\gamma^{BD} \tilde{V}_\gamma^D, \tag{68}$$

$$\hat{F}_\beta^{BD} \tilde{V}_\beta^B > \hat{F}_\gamma^{BD} \tilde{V}_\gamma^B, \tag{69}$$

$$\hat{F}_\beta^{BD} \tilde{V}_\beta^D > \hat{F}_\gamma^{BD} \tilde{V}_\gamma^B. \tag{70}$$

Now let us suppose that $\operatorname{sgn} \tilde{V}_\beta^B \neq \operatorname{sgn} \tilde{V}_\beta^D$ (i.e., $\sigma_1 = -1$). This means that the left-hand side of either Eq. (67) or (68) must be negative. The right-hand side of these two relations is identical, and must be less than some negative number:

$$F_\gamma^{BD} \tilde{V}_\gamma^D < 0. \tag{71}$$

Similar reasoning with Eqs. (69) and (70) leads to

$$F_\gamma^{BD} \tilde{V}_\gamma^B < 0. \tag{72}$$

These two inequalities require that V_γ^D and V_γ^B have the same sign. Thus we have shown that

$$\operatorname{sgn} \tilde{V}_\beta^B \neq \operatorname{sgn} \tilde{V}_\beta^D \Rightarrow \operatorname{sgn} \tilde{V}_\gamma^B = \operatorname{sgn} \tilde{V}_\gamma^D. \tag{73}$$

In the same way, one can begin with the assumption that V_γ^D and V_γ^B have opposite signs, and obtain

$$\operatorname{sgn} \tilde{V}_\gamma^B \neq \operatorname{sgn} \tilde{V}_\gamma^D \Rightarrow \operatorname{sgn} \tilde{V}_\beta^B = \operatorname{sgn} \tilde{V}_\beta^D. \tag{74}$$

Now let us use these results to determine the consistency of state D. Using the information given in Table III, Eq. (73) becomes

$$\sigma_1 \neq 1 \Rightarrow \sigma_2 = 1. \tag{75}$$

Recall that $\sigma_1 = -1$ means that state C is inconsistent, and $\sigma_1 = -1$, $\sigma_2 = 1$ is the condition required for D to be consistent. Thus the consistent state always exists.

But Eqs. (73) and (74) also have a much more profound implication. When moving from state B to state D, the sign of $\tilde{V}$ at both β and γ cannot change. Note that this reasoning holds even if there are other sliding contacts, as long as they remain sliding in all four states A, B, C, and D. Thus we have a second special case of the status change theorem.

Two-contact status change theorem. If two states A and B differ only in the status of two contacts β and γ, then $\operatorname{sgn} \tilde{V}_\beta^A = \operatorname{sgn} \tilde{V}_\beta^B$ or $\operatorname{sgn} \tilde{V}_\gamma^A = \operatorname{sgn} \tilde{V}_\gamma^B$, meaning that the consistency at β or γ (or both) must change.

C. Many sliding contacts

In this section, we prove the status change theorem. Let us restate it in this way.

Status change theorem. If two states A and B differ only in the status of n contacts $\beta_1, \ldots, \beta_n$, then for at least one contact β_i, we have $\operatorname{sgn} \tilde{V}_{\beta_i}^A = \operatorname{sgn} \tilde{V}_{\beta_i}^B$, implying that the consistency at β_i must also change.

Recall that in Sec. II D, we showed that this statement is sufficient to prove that the globally consistent state exists and is unique. We now establish this theorem.

Consider two different states A and B. The contacts in $\mathbb{A}$ are sliding in A but not in B, and the contacts in $\mathbb{B}$ are sliding in B but not in A. Let $\mathbb{S}$ contain contacts that are sliding in both states. We want to show that

$$\operatorname{sgn} \tilde{V}_\alpha^A = \operatorname{sgn} \tilde{V}_\alpha^B \quad \text{for at least one } \alpha \in \mathbb{A} \cup \mathbb{B}. \tag{76}$$

We will begin by assuming the contrary:

$$\operatorname{sgn} \tilde{V}_\alpha^A \neq \operatorname{sgn} \tilde{V}_\alpha^B \quad \text{for all } \alpha \in \mathbb{A} \cup \mathbb{B}, \tag{77}$$

and show that this leads to a contradiction.

The independent load condition Eq. (58) implies

$$Q(\mathbf{v}^A - \mathbf{v}^B, \mathbb{S}) = \sum_{\alpha \in \mathbb{A}} \hat{F}_\alpha^{AB} \tilde{V}_\alpha^A - \sum_{\alpha \in \mathbb{B}} \hat{F}_\alpha^{AB} \tilde{V}_\alpha^B. \tag{78}$$

Now let us assume stability for a third state C. The following contacts will be sliding in C: (1) All contacts in $\mathbb{S}$, who are sliding in both states A and B. (2) Some contacts that are sliding in A but not in B. Let $\mathbb{A}' \subset \mathbb{A}$ denote these contacts. (3) Some contacts that are sliding in B but not in A. Let $\mathbb{B}' \subset \mathbb{B}$ denote these contacts. Table IV summarizes this information.

TABLE IV. States for the proof of the status change theorem. Two states A and B are considered. For the proof, a third state C is constructed. Contacts in $\mathbb{A}$ are sliding in A, but not in B, and contacts in $\mathbb{B}$ are sliding in B, but not in A. Sets $\mathbb{A}' \subset \mathbb{A}$ and $\mathbb{B}' \subset \mathbb{B}$ are sliding in state C. (Here S=sliding, NS=nonsliding.)

State	$\mathbb{A}\backslash\mathbb{A}'$	$\mathbb{A}'$	$\mathbb{B}\backslash\mathbb{B}'$	$\mathbb{B}'$	$\mathbb{S}$	All others
A	S	S	NS	NS	S	NS
B	NS	NS	S	S	S	NS
C	NS	S	NS	S	S	NS

Let $\mathbb{C}=\mathbb{A}\cup\mathbb{B}$ and $\mathbb{C}'=\mathbb{A}'\cup\mathbb{B}'$. Note that $\mathbb{C}'$ can be any subset of $\mathbb{C}$. The stability condition for state C is

$$Q(\mathbf{v}^A-\mathbf{v}^B,\mathbb{S}\cup\mathbb{C}')>0,$$

$$Q(\mathbf{v}^A-\mathbf{v}^B,\mathbb{S})+\sum_{\alpha\in\mathbb{C}'}\hat{F}_\alpha^{AB}(\tilde{V}_\alpha^B-\tilde{V}_\alpha^A)>0. \tag{79}$$

Combining this with Eq. (78) yields

$$\sum_{\alpha\in\mathbb{A}'}\hat{F}_\alpha^{AB}\tilde{V}_\alpha^B+\sum_{\alpha\in\mathbb{A}\backslash\mathbb{A}'}\hat{F}_\alpha^{AB}\tilde{V}_\alpha^A-\sum_{\alpha\in\mathbb{B}'}\hat{F}_\alpha^{AB}\tilde{V}_\alpha^A-\sum_{\alpha\in\mathbb{B}\backslash\mathbb{B}'}\hat{F}_\alpha^{AB}\tilde{V}_\alpha^B>0. \tag{80}$$

Note that $\mathbb{A}'$ and $\mathbb{B}'$ are arbitrary, so there is a large number of such relations.

To write these relations in a more compact form, we define

$$\phi_\alpha^A=\begin{cases}F^{AB}\tilde{V}_\alpha^A & \text{for } \alpha\in\mathbb{A},\\ -F^{AB}\tilde{V}_\alpha^A & \text{for } \alpha\in\mathbb{B},\end{cases} \tag{81}$$

with an analogous definition for ϕ_α^B. Now the relations Eq. (80) can be written

$$\sum_{\alpha\in\mathbb{C}'}\phi_\alpha^B+\sum_{\alpha\in\mathbb{C}\backslash\mathbb{C}'}\phi_\alpha^A>0. \tag{82}$$

And the hypothesis Eq. (77) becomes

$$\operatorname{sgn}\phi_\alpha^A\neq\operatorname{sgn}\phi_\alpha^B \quad \text{for all } \alpha\in\mathbb{C}, \tag{83}$$

To show that Eq. (83) leads to a contradiction, it suffices to show that

$$\sum_{\alpha\in\mathbb{C}_i'}\phi_\alpha^B+\sum_{\alpha\in\mathbb{C}_i\backslash\mathbb{C}_i'}\phi_\alpha^A>0 \quad \text{for } 1\leqslant i\leqslant n, \tag{84}$$

where n is the number of elements in $\mathbb{C}$, and $\mathbb{C}_i$ is a subset of $\mathbb{C}$ that contains exactly i elements, and $\mathbb{C}_i'$ is any subset of $\mathbb{C}_i$.

The case $i=1$ of Eq. (84) contradicts the hypothesis Eq. (83). To see this, let $\mathbb{C}_1=\{\alpha\}$. Choosing $\mathbb{C}_1'=\varnothing$ in Eq. (84) leads to $\phi_\alpha^A>0$, and choosing $\mathbb{C}_1'=\{\alpha\}$ leads to $\phi_\alpha^B>0$. Thus ϕ_α^A and ϕ_α^B do not have opposite signs, as assumed in Eq. (83). This means that Eq. (77) is false, and Eq. (76) must be true. Equation (76) is equivalent to the status change theorem.

We now show Eq. (84) by induction, beginning with $i=n$ and proceeding to $i=1$. The case $i=n$ is trivial, since $\mathbb{C}_n=\mathbb{C}$. In this case, Eqs. (82) and (84) are identical.

Now let show that if Eq. (84) holds for $i+1$, then it holds for i also. Choose a contact β such that $\beta\notin\mathbb{C}_i$, but $\beta\in\mathbb{C}$. By the hypothesis, Eq. (84) holds for $\mathbb{C}_{i+1}=\mathbb{C}_i\cup\{\beta\}$. Next, we make two different choices for $\mathbb{C}_{i+1}'$ and apply Eq. (84). First we choose $\mathbb{C}_{i+1}'=\mathbb{C}_i'$ and obtain

$$\sum_{\alpha\in\mathbb{C}_i'}\phi_\alpha^B+\sum_{\alpha\in\mathbb{C}_i\backslash\mathbb{C}_i'}\phi_\alpha^A>-\phi_\beta^A, \tag{85}$$

and next we choose $\mathbb{C}_{i+1}'=\mathbb{C}_i'\cup\{\beta\}$:

$$\sum_{\alpha\in\mathbb{C}_i'}\phi_\alpha^B+\sum_{\alpha\in\mathbb{C}_i\backslash\mathbb{C}_i'}\phi_\alpha^A>-\phi_\beta^B. \tag{86}$$

Note the parallel between these conditions and Eqs. (67)–(70). By Eq. (83) either ϕ_β^A or ϕ_β^B is negative. Therefore, the only way for both of these inequalities to hold is if the sums are positive:

$$\sum_{\alpha\in\mathbb{C}_i'}\phi_\alpha^B+\sum_{\alpha\in\mathbb{C}_i\backslash\mathbb{C}_i'}\phi_\alpha^A>0. \tag{87}$$

This completes the induction step, and thus the proof.

V. DISCUSSION AND CONCLUSION

A. How restrictive are the assumptions?

This work considered only circular particles. The conclusions are probably not modified if other shapes are considered. The particle shape most strongly affects the geometric stiffness, which is neglected because of the quasirigid assumption. The moment of inertia plays only a small role, because it is eliminated in the quasistatic approximation. To accommodate particles of different shapes, the particle radius r in Eq. (13) must depend on the contact, and the torque may also depend on the normal force. This requires modifying the matrix $\mathbf{c}$. But none of these should alter our considerations of mechanism indeterminacy, nor alter the two premises of the proof used to resolve contact status indeterminacy.

Another assumption that seems quite restrictive is the use of the linear force law in deriving the stiffness matrix. This is no restriction, because this paper revolves around the question of what occurs at one point in time, when the system must adjust the status of the contacts. Therefore, one could always linearize the force law around the positions of the particles.

A second assumption is that all possible states must be stable. This assumption is reasonable because if there is an unstable state, the packing will probably collapse, rendering the question of uniqueness irrelevant. The state that is most likely to be unstable is the one where all M_c critical contacts are sliding. This is so, because the contributions of sliding contacts to the quadratic form are on the average negative [26]. Furthermore, the only way to obtain instability is for negative contributions of the sliding contacts to the quadratic form outweigh the positive contributions of the nonsliding contacts [see discussion of Eq. (49)]. Now, the packing is always close to this state, because when a contact becomes nonsliding it leaves the boundary of the Coulomb cone and

becomes noncritical. Therefore, when the assumption of stability for all possible states is violated, we expect the packing to yield.

Furthermore, if there are vibrations in the system, they will be governed by Eq. (38). These vibrations will cause the relative motion at each contact to fluctuate. At critical contacts, the contact status will therefore switch between nonsliding and sliding. Thus the packing will sample many different possible states, and if it finds an unstable one, it may collapse. Thus all states must be stable in order to guarantee that the vibrations will damp out, meaning that this is a necessary condition to obtain quasistatic balance.

Finally, let us remark that theory presented in Sec. III requires some modifications to deal with opening and closing contacts. To account for a contact that opens, it is necessary to introduce the status "open," and allow D_n to become positive. Another problem is presented by contacts that are initially open but may later close. In the theory, the particle deformations are assumed to be infinitely small, so that no two particles separated by a finite distance will ever touch. This may lead to the omission of important effects when the particle separations are very small, such as in a regular packing of almost monodisperse spheres.

B. Implications of the result

This work suggests that the stiffness matrix together with the principle of minimum sliding form a complete description of quasistatic granular material. Since the globally consistent state always exists and is unique, there is no need to appeal to other processes that have been left out of the model to decide between various possible states. Instead of rather brutally setting the particle displacements to zero, as is done in various stress-only approaches to granular matter, one should consider taking the quasirigid limit, where the stiffness of the particles diverges, and the displacements become infinitesimally small, but are not set to zero. Taking this limit leads to the stiffness matrix approach discussed in this paper. This work supports the conjecture that the quasirigid limit preserves all the necessary physics needed to describe the quasistatic behavior.

Of course, there remain some open questions. For example, the question of opening and closing contacts has not been dealt with. When a contact reaches the apex of the Coulomb cone, it can then go into four different states. It can become nonsliding, and move into the interior of the Coulomb cone. Then, there are two distinct sliding states—each one corresponding to a different side of the Coulomb cone. Finally, the contact can open. Are these four states mutually exclusive, as we have shown to be the case for the sliding and nonsliding states in this paper?

This work should also encourage the use of numerical methods based on the stiffness matrix. The problem faced by these methods is, of course, to find the globally consistent contact status. This work shows that such a state always exists, and is unique. Thus any way to find the state is acceptable. Furthermore, perhaps it is possible to use the results of this paper to design intelligent strategies for finding the globally consistent state.

ACKNOWLEDGMENTS

We thank Jean-Noël Roux for drawing our attention to contact status indeterminacy. We also thank Matthew Kuhn and Katalin Bagi for making their manuscripts available. We acknowledge the support of the European DIGA Project No. HPRN-CT-2002-00220, and the Deutsche Forschungsgemeinschaft through project GEP-HE-2732/8-1.

[1] *Physics of Dry Granular Media*, edited by H. J. Herrmann, J.-P. Hovi, and S. Luding (Kluwer, Dordrecht, 1998).
[2] S. N. Coppersmith, C.-h. Liu, S. Majumdar, O. Narayan, and T. A. Witten, Phys. Rev. E **53**, 4673 (1996).
[3] J. P. Bouchaud, P. Claudin, M. E. Cates, and J. P. Wittmer, in *Physics of Dry Granular Media* (Ref. [1]).
[4] P. Claudin and J.-P. Bouchaud, in *Physics of Dry Granular Media* (Ref. [1]).
[5] M. E. Cates, J. P. Wittmer, J.-P. Bouchaud, and P. Claudin, Philos. Trans. R. Soc. London, Ser. A **356**, 2535 (1998).
[6] S. B. Savage, in *Physics of Dry Granular Media* (Ref. [1]).
[7] F. Cantelaube, A. K. Didwania, and J. D. Goddard, in *Physics of Dry Granular Media* (Ref. [1]).
[8] A. K. Didwania, F. Cantelaube, and J. D. Goddard, Proc. R. Soc. London, Ser. A **456**, 2569 (2000).
[9] F. Radjai, L. Brendel, and S. Roux, Phys. Rev. E **54**, 861 (1996).
[10] T. Unger, J. Kertész, and D. E. Wolf, Phys. Rev. Lett. **94**, 178001 (2005).
[11] J. H. Snoeijer, T. J. H. Vlugt, M. van Hecke, and W. van Saarloos, Phys. Rev. Lett. **92**, 054302 (2004).
[12] S. McNamara and H. Herrmann, Phys. Rev. E **70**, 061303 (2004).
[13] J. J. Moreau, in *Novel Approaches in Civil Engineering*, edited by M. Frémond and F. Maceri, Lecture Notes in Applied and Computational Mechanics Vol. 14 (Springer, Berlin, 2004), p. 1.
[14] Y. Kishino, in *Micromechanics of Granular Materials*, edited by M. Satake and J. T. Jenkins (Elsevier, Amsterdam, 1988).
[15] K. Bagi, Mech. Mater. **16**, 101 (1993).
[16] X. Zhuang, A. K. Didwania, and J. D. Goddard, J. Comput. Phys. **121**, 331 (1995).
[17] Y. Kishino, H. Akaizawa, and K. Kaneko, in *Powders and Grains 2001*, edited by Y. Kishino (Balkema, Rotterdam, 2001).
[18] G. Oron and H. J. Herrmann, in *Physics of Dry Granular Media* (Ref. [1]).
[19] J.-N. Roux, Phys. Rev. E **61**, 6802 (2000).
[20] G. Combe and J.-N. Roux, Phys. Rev. Lett. **85**, 3628 (2000).
[21] J.-N. Roux and G. Combe, C. R. Phys. **3**, 131 (2002).
[22] G. Combe, Ph.D. thesis, Labaratoire central des Ponts et Chaussées, 2002.

[23] C. Miehe and J. Dettmar, Comput. Methods Appl. Mech. Eng. **193**, 225 (2003).
[24] S. McNamara, R. García-Rojo, and H. Herrmann, Phys. Rev. E **72**, 021304 (2005).
[25] M. R. Kuhn and C. S. Ching, Int. J. Solids Struct. **43**, 6026 (2006).
[26] S. McNamara, R. García-Rojo, and H. J. Herrmann, in *Powders & Grains 2005*, edited by R. García-Rojo, H. J. Herrmann, and S. McNamara (Balkema, Leiden, 2005).
[27] K. Bagi, Granular Matter (to be published).